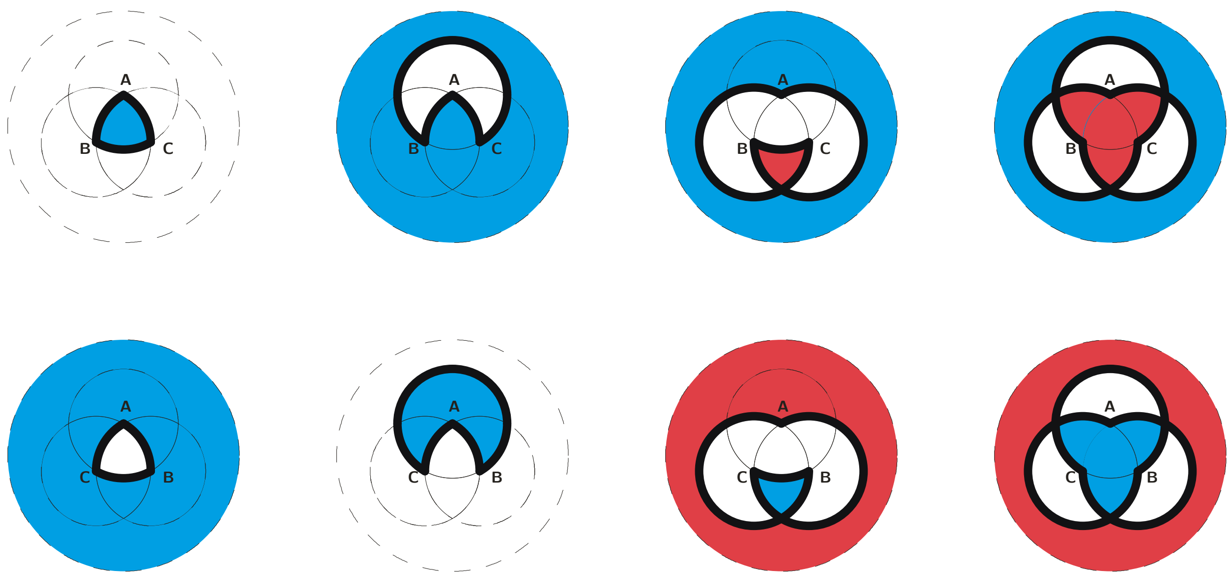
Generalized Spherical Triangles (GST).

Abstract. The purpose is to introduce spherical triangles with one ore more angles or sides greater than pi.
A signed theory extends with a little modification the connection between area and spherical excess , and also facilitates a classification of triangles. Unsigned entities is easily deductible with standard spherical trigonometry.

In $\mathbb{R}^3$ set of tree unit vectors **a**, **b**, **c** not in a plane determines a spherical triangle ΔABC, and also an orientation $\text{sgn}(ABC)$ as the sign of the determinant $[\![[\![\mathbf{a}]\!] [\![\mathbf{b}]\!] [\![\mathbf{c}]\!]]\!]$ where $[\![\mathbf{x}]\!]$ is the coordinates of a vector **x** is relative to the standard basis.
A generalized spherical triangle is given by such three points A,B,C and three oriented great-circle segments $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CA}$ each directed from the first point to the second.
As each segment can be short or long, i.e. in $(0, \pi)$ or $(\pi, 2\pi)$, there are eight GST's with given vertices A, B, C. A long segment is indicated by an arc as in $\text{GST}_+ (A \frown BC \frown)$ meaning a positively oriented GST with long segments $\overset{\frown}{AB}$ and $\overset{\frown}{CA}$ and a short segment $\overset{\frown}{BC}$.
The angle $\measuredangle A$ (or just A) of a GST is determined as the angle in $(0, 2\pi)$ from the direction of $\overset{\frown}{AB}$ at A to the opposite direction of $\overset{\frown}{CA}$ at A.
The three great-circles that contain the sides of the GST divides the sphere into eight elementary GST's and a orientation of a part of these connected along the sides are possible only in one way, if it shall be consistent with the original GST.
The results are illustrated by the figure and given in TableForm.



The stereographic images illustrates GST types 1, 2 a, 3 a, 4, 5, 6 a, 7 a, 8.
Positively oriented elementary triangles are blue, negatively oriented red.
The closed triangle curve has one double-point for types #3, #7, three for #4, #8, and otherwise zero.

On the figure the colored part is considered as the inside part of the triangle with blue positive and red marking negative area density.
The case $\text{sgn}(ABC) > 0$ is illustrated in the first row and the case $\text{sgn}(ABC) < 0$ in the second row.
The first row has 4 types corresponding to some of the first 8 cases mentioned detailed in the table.

Likewise corresponds the second row to some of the remaining 8 cases in the table.
The full table may show the be practical in connection with calculations.

#	**index**	**notation**	a	b	c	A	B	C	T	E
1	(0, 0)	$\text{GST}_+(ABC)$	a_0	b_0	c_0	A_0	B_0	C_0	T_0	T_0
2 a	(1, 2)	$\text{GST}_+(AB\frown C)$	$2\pi - a_0$	b_0	c_0	A_0	$\pi + B_0$	$\pi + C_0$	$2\pi + T_0$	$2\pi + T_0$
2 b	(1, 2)	$\text{GST}_+(A\frown BC)$	a_0	b_0	$2\pi - c_0$	$\pi + A_0$	$\pi + B_0$	C_0	$2\pi + T_0$	$2\pi + T_0$
2 c	(1, 2)	$\text{GST}_+(ABC\frown)$	a_0	$2\pi - b_0$	c_0	$\pi + A_0$	B_0	$\pi + C_0$	$2\pi + T_0$	$2\pi + T_0$
3 a	(2, 2)	$\text{GST}_+(A\frown BC\frown)$	a_0	$2\pi - b_0$	$2\pi - c_0$	A_0	$\pi + B_0$	$\pi + C_0$	T_0	$2\pi + T_0$
3 b	(2, 2)	$\text{GST}_+(AB\frown C\frown)$	$2\pi - a_0$	$2\pi - b_0$	c_0	$\pi + A_0$	$\pi + B_0$	C_0	T_0	$2\pi + T_0$
3 c	(2, 2)	$\text{GST}_+(A\frown B\frown C)$	$2\pi - a_0$	b_0	$2\pi - c_0$	$\pi + A_0$	B_0	$\pi + C_0$	T_0	$2\pi + T_0$
4	(3, 0)	$\text{GST}_+(A\frown B\frown C\frown)$	$2\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	A_0	B_0	C_0	$T_0 - 2\pi$	T_0
5	(0, 3)	$\text{GST}_-(ABC)$	a_0	b_0	c_0	$2\pi - A_0$	$2\pi - B_0$	$2\pi - C_0$	$4\pi - T_0$	$4\pi - T_0$
6 a	(1, 1)	$\text{GST}_-(AB\frown C)$	$2\pi - a_0$	b_0	c_0	$2\pi - A_0$	$\pi - B_0$	$\pi - C_0$	$2\pi - T_0$	$2\pi - T_0$
6 b	(1, 1)	$\text{GST}_-(A\frown BC)$	a_0	b_0	$2\pi - c_0$	$\pi - A_0$	$\pi - B_0$	$2\pi - C_0$	$2\pi - T_0$	$2\pi - T_0$
6 c	(1, 1)	$\text{GST}_-(ABC\frown)$	a_0	$2\pi - b_0$	c_0	$\pi - A_0$	$2\pi - B_0$	$\pi - C_0$	$2\pi - T_0$	$2\pi - T_0$
7 a	(2, 1)	$\text{GST}_-(A\frown BC\frown)$	a_0	$2\pi - b_0$	$2\pi - c_0$	$2\pi - A_0$	$\pi - B_0$	$\pi - C_0$	$-T_0$	$2\pi - T_0$
7 b	(2, 1)	$\text{GST}_-(AB\frown C\frown)$	$2\pi - a_0$	$2\pi - b_0$	c_0	$\pi - A_0$	$\pi - B_0$	$2\pi - C_0$	$-T_0$	$2\pi - T_0$
7 c	(2, 1)	$\text{GST}_-(A\frown B\frown C)$	$2\pi - a_0$	b_0	$2\pi - c_0$	$\pi - A_0$	$2\pi - B_0$	$\pi - C_0$	$-T_0$	$2\pi - T_0$
8	(3, 3)	$\text{GST}_-(A\frown B\frown C\frown)$	$2\pi - a_0$	$2\pi - b_0$	$2\pi - c_0$	$2\pi - A_0$	$2\pi - B_0$	$2\pi - C_0$	$2\pi - T_0$	$4\pi - T_0$

The table is established as

1. Index is (# sides $> \pi$, # angles $> \pi$).
2. An angle is changed with π for each adjacent side changed between long and short.
3. A vertex angle u is changed to $2\pi - u$, if the triangle is mirrored.
4. T is calculated as sum of the signed areas of the signed elementary triangles.
5. The spherical excess E is calculated as $A + B + C - \pi$.

and imply obviously:

Corollary. Side index $\in \{1, 2\} \Leftrightarrow$ Angle index $\in \{1, 2\}$

Theorem. For a GST the area is given from spherical excess by the formula

$$T = E - \pi\,\delta \text{ with}$$

$\delta = 2$, if the number of long sides is ≥ 2, and otherwise $\delta = 0$.

Solving GST for three objects of A, B, C, a, b, c known is done by first calculating
objects corresponding to A_0, B_0, C_0, a_0, b_0, c_0.
Hence sides greater than π is subtracted from 2π. Two orientation cases are treated separately.
If orientation of the triangle is positive, then angles greater than π are subtracted π.
Otherwise angles greater than π is subtracted from 2π.
Then this usual spherical triangle case is solved to get A_0, B_0, C_0, a_0, b_0, c_0.
The side index determines the triangle type and the objects is calculated after the table.

Example1: Let $A = 1.3\,\pi$, $a = 1.4\,\pi$, $b = 2$.
In the positive case $A_0 = 1.3\,\pi - \pi = 0.3\,\pi$, $a_0 = 2\,\pi - a = 0.6\,\pi$, $b_0 = b$ and we find
$B_0 = 2.257256214$, $C_0 = 0.1595295613$, $c_0 = 0.1878465826$, $T_0 = 0.2176709175$.
#3c in the table is the only possibility in this positive case and
$B = B_0$, $C = \pi + C_0 = 3.301122215$, $c = 2\,\pi - c_0 = 6.095338725$, $T = T_0$.
In the negative case $A_0 = 2\,\pi - 1.3\,\pi = 0.7\,\pi$, $a_0 = 2\,\pi - a = 0.6\,\pi$, $b_0 = b$ and we find
$B_0 = 2.257256214$, $C_0 = 2.260798179$, $c_0 = 2.006311836$, $T_0 = 3.575576597$
#6a in the table is the only possibility among #5-8. Hence
$B = \pi - B_0 = 0.8843364399$, $C = \pi - C_0 = 0.8807944746$, $c = c_0$, $T = 2.707608711$

Example 2: Let $A = 1.8\,\pi$, $B = 1.6\,\pi$, $a = 0.9\,\pi$.
In the positive case $A_0 = 1.8\,\pi - \pi = 0.8\,\pi$, $B_0 = 1.6\,\pi - \pi = 0.6\,\pi$, $a_0 = a$ and we find 2 solutions
1) $C_0 = 1.338387952$, $b_0 = 2.617993878$, $c_0 = 0.5370419222$, $T_0 = 2.595025014$ and by #2b
$C = C_0$, $b = b_0$, $c = 2\,\pi - c_0 = 5.746143385$, $T = 2\,\pi + T_0 = 8.878210321$,
2) $C_0 = 2.461621775$, $b_0 = 0.5235987756$, $c_0 = 2.804691386$, $T_0 = 3.718258836$ and by #2b
$C = C_0$, $b = b_0$, $c = 2\,\pi - c_0 = 3.478493921$, $T = 2\,\pi + T_0 = 10.00144414$.
The negative case $A_0 = 2\,\pi - 1.8\,\pi = 0.2\,\pi$, $B_0 = 2\,\pi - 1.6\,\pi = 0.4\,\pi$, $a_0 = a$ has 0 solutions.
[It can be proved, if the positive/negative case has exactly two elementary solutions, then
the opposite signed case is without solutions.]